Very unbalanced Chess Positions

Marco Ripà

sPIqr Society, World Intelligence Network
Rome, Italy
e-mail: marcokrt1984@yahoo.it

Abstract: In this paper, we solve a few optimization problems regarding chess. It is well known that chess pieces have different values, and the standard scoring criteria assign 9 points to the queen (Q=9), 5 points to the rook (R=5), 3 points for both bishop and knight (B=3=N), while a pawn is evaluated only 1 point (P=1). Since kings cannot be taken, their value is undetermined; hence, we assume K=0. Our goal is to maximize the score gap between the White and the Black, under different constraints, letting the disadvantaged player checkmate his opponent, or end the game as a draw by stalemate.

Keywords: Board games, Chess, 2-Person games, Two-Player games, Optimization problem, Checkmate, Stalemate, Zero-sum game, Pieces value, Game theory.

2010 Mathematics Subject Classification: 91A05, 91A24.

1 Introduction

Chess is one of the oldest finite two-player zero-sum board games.
FIDE standard rules state when a match can be declared as a draw, and one of them involves stalemate (see FIDE Laws of Chess, Article 9, rule 9.6 [3]).
Stalemate occurs when the player on move is not able to perform any legal move, and at the same time is not threatened by check. Thus, stalemate can be forced by both players: we define it as "active" if the player moves without allowing his opponent to perform any legal move, and we call it "passive" if the player has no legal move available (it is an active stalemate for his opponent).
In the present paper we study three different maximization problems, taking into account the final score on the board for both players. The aforementioned score is based on the standard chess piece relative value system [1], resumed in Potulates 1-5.

Postulate 1.
Pawn := (P) = 1 point;
Knight := (N) = 3 points;
Bishop := (B) = 3 points;
Rook := (R) = 5 points;
Queen := (Q) = 9 points;
King := (K) = 0 points.

Postulate 2. In the whole game (at any move), each player has a total score given by the sum of the values of all his pieces on the board.

Postulate 3. The final score of each player is given by the total score by Postulate 2, when the last legal move of the match has been played.

Postulate 4. The final move of a game is the one that produces checkmate or stalemate.

Postulate 5. The final score difference between player White and player Black is given by

$$final\ score\ White\ - final\ score\ Black, \qquad (1)$$

while, at any intermediate move of the game, the score difference between player White and player Black is

$$score\ White\ (current\ position) - score\ Black\ (current\ position). \qquad (2)$$

Obviously, the result from (2) is a non negative integer (natural numbers including zero) if and only if the White has no material disadvantage on the board at the current (half) move of the game. We add the following coinstraint to the 5 postulates stated above.

Constraint 1. Considering any move of the game, for both players, $score\,\mathrm{W} - \mathrm{score\ B} \geq 0$ is (always) true.

The problems which we are going to solve in Section 2, under Constraint 1, are as follows:

Problem 1. Which is the maximum value of (1) to let the Black win the game?

Problem 2. Which is the maximum value of (1) to let the Black end the game with a passive stalemate?

Problem 3. Which is the maximum value of (1) to let the Black end the game with an active stalemate?

Problem 4. Which is the maximum value of $final\ score\ White$ to let the Black end the game with an active stalemate?

2 Maximizing the negative score difference without losing the game

We answer the questions introduced by Problems 1-4 (Section 1) using the well known PGN notation (Portable Game Notation [5]) in order to prove that any given final FEN (Forsyth-Edwards Notation [4]) position is legit (no matter if the same score can be achieved through a smaller number of moves and/or reaching a different position).

Problem 1. Which is the maximum value of (1) to let the Black win the game?

Lemma 1. A score of 103 points is the maximum score attainable in a chess match.

Proof of Lemma 1.
At the beginning of the game, there are 8 pawns on the board for each player: assuming that all of them are promoted to a queen and no other piece is lost, the total score is given by

$$Maximum\ score = 9 \cdot Q + 2 \cdot R + 2 \cdot B + 2 \cdot N = 9 \cdot 9 + 2 \cdot 5 + 2 \cdot 3 + 2 \cdot 3 = 103. \quad (3)$$

Theorem 1. The maximum value of (1) to let the Black win the game is 102.

Proof of Theorem 1.
We invoke Lemma 1. Since a king cannot checkmate the other king by himself, he needs at least one pawn of the same color in order to win the match. It follows that the maximum value of (1) to let the Black win the game is given by $Maximum\ score$ minus P=1.
Therefore, we complete the proof showing that a 102 score is really achievable in a standard game with only legal moves (FIDE rules), ending the match in a win for the Black.

PGN (0 - 1)
1. a4 b5 2. axb5 a6 3. bxa6 Bb7 4. axb7 c6 5. bxa8=Q Qc8 6. b4 Qd8 7. b5 Qc7 8. b6 Qd8 9. b7 Na6 10. b8=Q c5 11. d4 f6 12. dxc5 Kf7 13. c6 Ke8 14. c7 Kf7 15. c8=Q Ke8 16. c4 Kf7 17. c5 Ke8 18. c6 Kf7 19. c7 Ke8 20. Qcxa6 Kf7 21. c8=Q Ke8 22. e4 Kf7 23. e5 Ke8 24. e6 Nh6 25. exd7+ Kf7 26. f4 g5 27. fxg5 Nf5 28. h4 h6 29. gxh6 Rg8 30. h7 Rg7 31. h8=Q Kg6 32. Kd2 Qe8 33. g4 Kf7 34. Kc3 Kg6 35. Kc4 Kf7 36. Kd5 Kg6 37. Na3 Kf7 38. Nc4 Kg6 39. dxe8=Q+ Rf7 40. Ba3 Bh6 41. Bd6 Bg7 42. h5+ Kg5 43. h6 Rf8 44. Qc5 Rg8 45. Qec6 Kg6 46. g5 Bf8 47. gxf6 Bg7 48. h7 Nd4 49. fxg7 Nc2 50. hxg8=Q Nd4 51. Qgd8 Kf5 52. g8=Q Nc2 53. Qg2 Nd4 54. Qhxd4 e6# □

The final position is shown in figure 1
(FEN: QQ1Q4/8/Q1QBp3/2QK1k2/2NQ4/8/6Q1/R2Q1BNR w - 0 55).

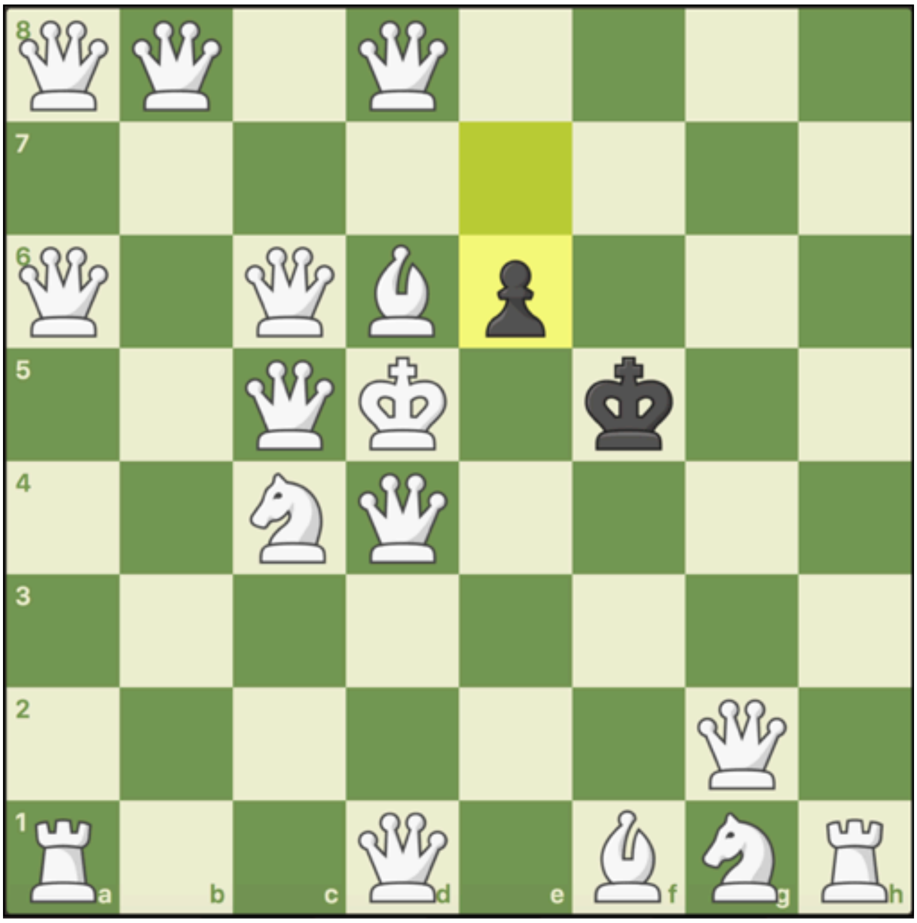

Figure 1. Final position of a Black victory, despite of the 102 points difference.

It is trivial to show how it is possible to change a few moves of the previous game to get an even 100 points final score difference and a loss for the White. As an example, let us modify moves 51 (et seq.) in order to checkmate the white king (103 points final score for the White) with one knight:

PGN (0 - 1)
1. a4 b5 2. axb5 a6 3. bxa6 Bb7 4. axb7 c6 5. bxa8=Q Qc8 6. b4 Qd8 7. b5 Qc7 8. b6 Qd8 9. b7 Na6 10. b8=Q c5 11. d4 f6 12. dxc5 Kf7 13. c6 Ke8 14. c7 Kf7 15. c8=Q Ke8 16. c4 Kf7 17. c5 Ke8 18. c6 Kf7 19. c7 Ke8 20. Qcxa6 Kf7 21. c8=Q Ke8 22. e4 Kf7 23. e5 Ke8 24. e6 Nh6 25. exd7+ Kf7 26. f4 g5 27. fxg5 Nf5 28. h4 h6 29. gxh6 Rg8 30. h7 Rg7 31. h8=Q Kg6 32. Kd2 Qe8 33. g4 Kf7 34. Kc3 Kg6 35. Kc4 Kf7 36. Kd5 Kg6 37. Na3 Kf7 38. Nc4 Kg6 39. dxe8=Q+ Rf7 40. Ba3 Bh6 41. Bd6 Bg7 42. h5+ Kg5 43. h6 Rf8 44. Qc5 Rg8 45. Qec6 Kg6 46. g5 Bf8 47. gxf6 Bg7 48. h7 Nd4 49. fxg7 Nc2 50. hxg8=Q Nd4 **51. Bxe7+ Kf5 52. Qgd8 Nb3 53. g8=Q Nc1 54. Rh4 Na2 55. Rd4 Nc1 56. Bd6 Na2 57. Qhh4 Nc3#**

Problem 2. Which is the maximum value of (1) to let the Black end the game with a passive stalemate?

Theorem 2. The maximum value of (1) to let the Black end the game with a passive stalemate is 103 points.

Proof of Theorem 2.
Since Lemma 1 states that the maximum score achievable is 103 points, then we prove that 103 points is a valid score when an active stalemate by the White occurs [6].

PGN (1/2 - 1/2)
1. a4 b5 2. axb5 a6 3. bxa6 Bb7 4. axb7 c6 5. bxa8=Q d5 6. c4 e6 7. cxd5 Qd7 8. dxc6 Be7 9. c7 Kf8 10. c8=Q+ Qe8 11. b4 Nc6 12. b5 Nd8 13. b6 Nc6 14. b7 Nd8 15. b8=Q h5 16. d4 g6 17. d5 Rh7 18. d6 Rh8 19. d7 Rh7 20. e4 Nc6 21. d8=Q f5 22. exf5 Nh6 23. fxe6 Bf6 24. e7+ Kg8 25. f4 Qf8 26. e8=Q Bg7 27. f5 Kh8 28. f6 Qg8 29. f7 g5 30. f8=Q Nf7 31. h4 Nfe5 32. hxg5 Bf6 33. gxf6 Rg7 34. f7 Kh7 35. Qfa3 Qh8 36. f8=Q Rg5 37. g4 Qg7 38. gxh5 Qg6 39. Qfb4 Kg7 40. Qcxc6 Nf7 41. Qee2 Nd6 42. h6+ Kf7 43. h7 Rd5 44. Qcxd6 Kg7 45. Q6xd5 Qd6 46. Q5xd6 Kf7 47. h8=Q □

The final position is shown in figure 2
(FEN: QQ1Q3Q/5k2/3Q4/8/1Q6/Q7/4Q3/RNBQKBNR b KQ - 0 47).

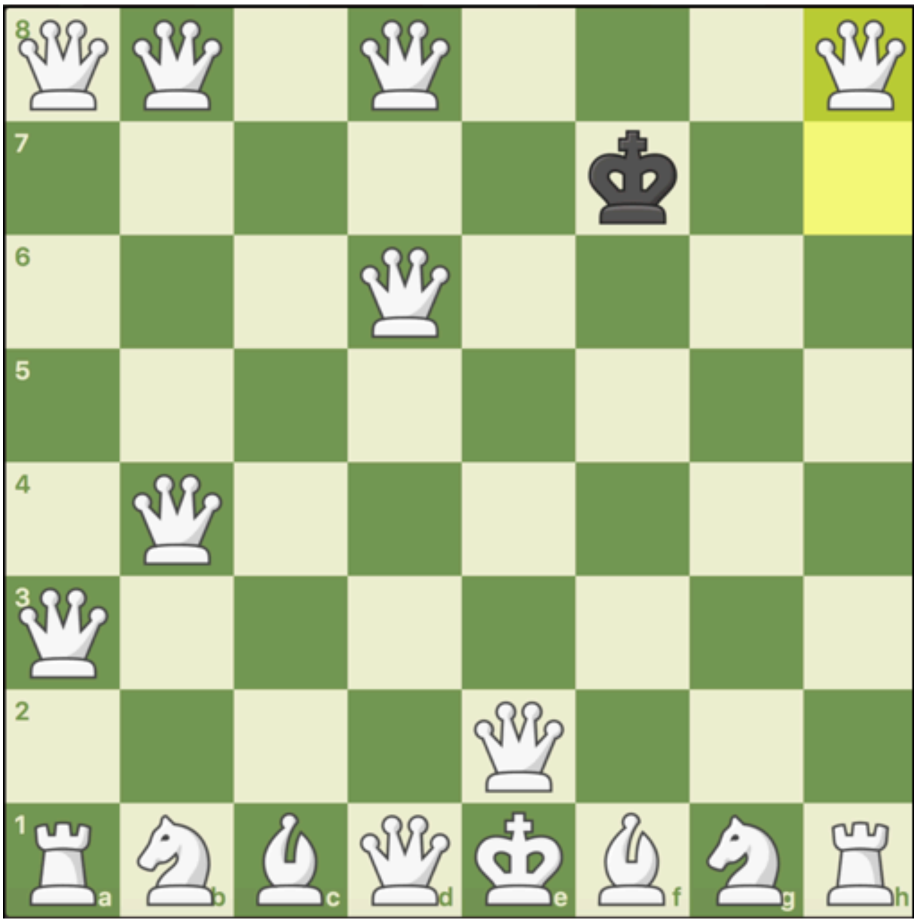

Figure 2. Final position of a passive Black stalemate, despite of a 103 points difference.

Problem 3. Which is the maximum value of (1) to let the Black end the game with an active stalemate?

Lemma 2. If we force the match to end with an active Black stalemate, 81 is a lower bound of the highest possible value of (1).

Proof of Lemma 2.
The following game is legit according to standards FIDE rules, and it ends with an active Black stalemate plus a 81 points difference for the White.

PGN (1/2 - 1/2)
1. g4 f5 2. gxf5 g5 3. h4 g4 4. h5 Kf7 5. f6 Ke6 6. fxe7 Kf6 7. e8=Q g3 8. Nf3 g2 9. Rh4 g1=B 10. a4 b5 11. axb5 a6 12. bxa6 Bb7 13. axb7 Nc6 14. bxa8=Q Qe7 15. Qeb8 Qd6 16. e4 Kg7 17. e5 Qd5 18. e6 Qe5+ 19. Qe2 Qf6 20. exd7 Qf7 21. d8=Q Nce7 22. Qdxc7 Nf6 23. Qea6 Nfd5 24. Qac8 Nf6 25. Ne5 Ned5 26. f4 Kg8 27. Nc6 Ng4 28. c4 Ne5 29. fxe5 Qf6 30. e6 Qf7 31. e7 Qf6 32. e8=Q Qf7 33. Nd4 Qe7+ 34. Kd1 Qf7 35. Qea4 Qg7 36. Q4a7 Qf7 37. c5 Qe6 38. Qd6 Qe7 39. c6 Qf7 40. c7 Qe7 41. Qcb7 Qf7

42. c8=Q Qe7 43. Qdc7 Qf7 44. Nf5 Nf6 45. d4 Ne4 46. d5 Bc5 47. Ne7+ Kg7 48. Bg5 Qg6 49. Nc6+ Qf7 50. Bd8 Nf6 51. Re4 Nd7 52. Re7 Kg8 53. Rxd7 Kg7 54. Ne7 Bb4 55. Ra6 Ba3 56. Rb6 Bc5 57. Ba6 Ba3 58. b4 Bb2 59. b5 Ba3 60. Nd2 Bb4 61. Nb3 Qe6 62. h6+ Kf7 63. Kc2 Rg8 64. Kd3 Rg7 65. hxg7 Kf6 66. g8=Q Ba3 67. Qg1 h5 68. Qgc5 h4 69. Q5c6 Kf7 70. Kd4 Qh6 71. Nc5 Qf4+ 72. Qxf4+ Kg7 73. Qfd6 h3 74. Qf4 h2 75. Qxh2 Kf7 76. Qhc7 Kg7 77. Ke5 Kf7 78. Kd6 Kf6 □

The final position is shown in figure 3
(FEN: QQQB1b2/QQQRN3/BRQK1k2/1PNP4/8/b7/8/8 w - 7 79).

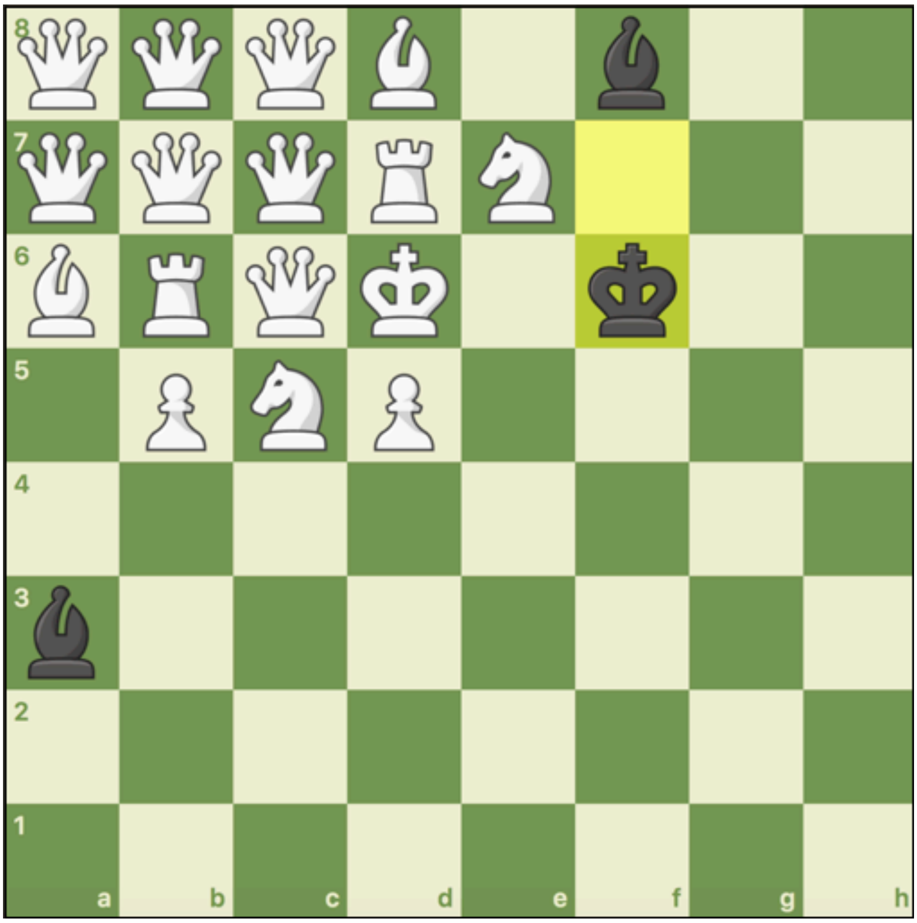

Figure 3. Final position of an active Black stalemate, despite of a 81 points difference.

We cannot formally prove that the result by Lemma 2 is the optimal one for Problem 3. Thus, we can only take it as a good lower bound.

Problem 4. Which is the maximum value of $final\ score\ White$ to let the Black end the game with an active stalemate?

Lemma 3. If we force the match to end with an active Black stalemate, 91 is a valid lower bound of the highest $final\ score\ White$ value attainable.

Proof of Lemma 3.
The following game is legit according to standards FIDE rules, and it ends with an active Black stalemate plus a 91 points final score for the White.

PGN (1/2 - 1/2)
1. g4 f5 2. gxf5 g5 3. a4 b5 4. axb5 a6 5. bxa6 Bb7 6. axb7 g4 7. bxa8=Q g3 8. Nf3 c6 9. h4 g2 10. h5 h6 11. c4 d5 12. cxd5 g1=B 13. dxc6 Kf7 14. c7 Qd7 15. cxb8=Q Qd8 16. d4 Qd7 17. Bxh6 Bg7 18. Bg5 e6 19. Qc2 Rh6 20. Qcc8 Rg6 21. d5 Qe7 22. d6 Qf6 23. d7 Qe5 24. d8=Q Qf6 25. Qda5 Qe5 26. Q5a7+ Qc7 27. Bd8 Rg5 28. h6 Rg6 29. h7 Rg5 30. h8=Q Rg6 31. e4 Rg5 32. Ba6 Bf8 33. Qh2 Qe7 34. Qhc7 Kg7 35. fxe6 Qf7 36. e7 Qf6 37. e8=Q+ Qf7 38. Nfd2 Rg6 39. f4 Rg2 40. e5 Rg6 41. e6 Rg5 42. Qeb5 Rg6 43. Q5b7 Rg4 44. e7 Nf6 45. e8=Q Nd5 46. Rh6 Nb6 47. Rxb6 Rg6 48. Qec6 Qd7 49. f5 Kh6 50. f6 Kg5 51. f7+ Kg4 52. Nc3 Bg7 53. f8=R Rf6 54. Re8 Kg5 55. Rd1 Bc5 56. Nde4+ Kf5 57. Rxd7 Ba3 58. b4 Bf8 59. Kd2 Kg4 60. Kd3 Kf5 61. Kd4 Kg4 62. Kd5 Kf5 63. Ree7 Kg4 64. Nc5 Rf1 65. Kd6 Rd1+ 66. Nd5 Kf5 67. b5 Rd2 ☐

The final position is shown in figure 4
(FEN: QQQB1b2/QQQRR3/BRQK4/1PNN1k2/8/b7/3r4/8 w - 1 68).

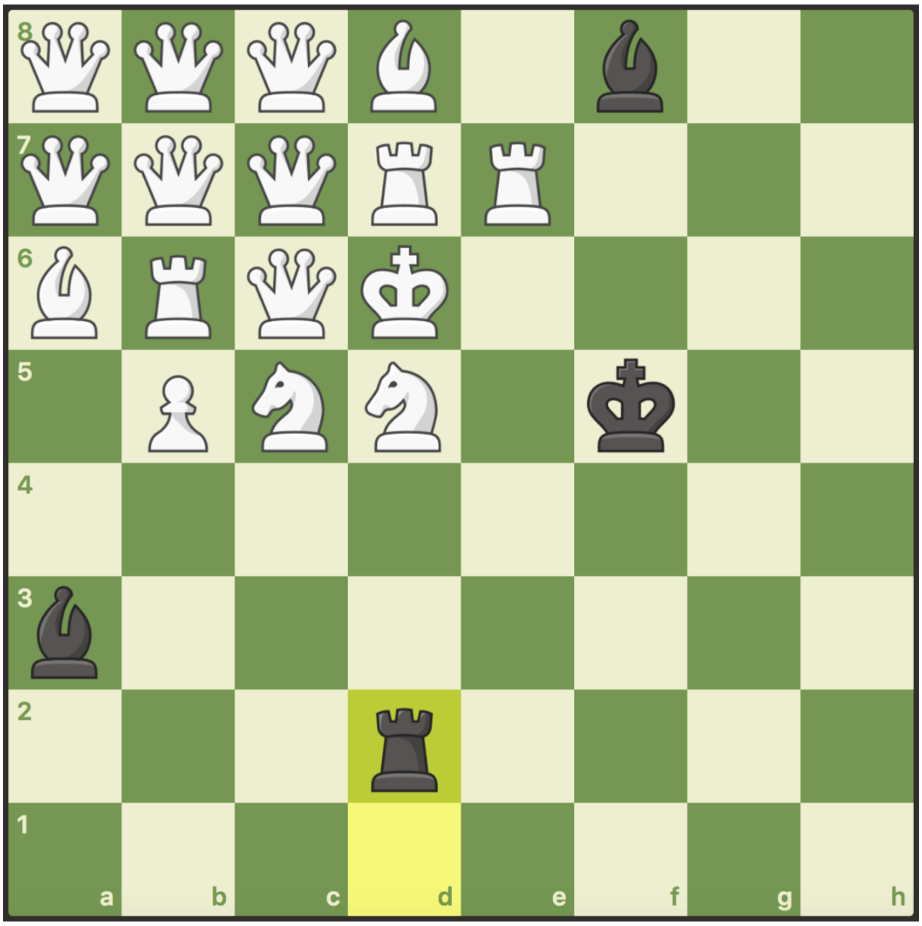

Figure 4. Final position of an active Black stalemate, despite of a 91 points score for his opponent.

As we previously pointed out, the 91 points score cannot be considered as an optimal solution: it merely shows a nontrivial lower bound for Problem 4.

3 Maximum total score achievable on the board for a generic draw

Since the standard FIDE rules allow a competition match to end in a draw even if a stalemate does not occur (see FIDE Laws of Chess, Article 9, rules 9.2&9.3 [3]), let us consider a generic game and a different problem.

Problem 5. Let us (provisionally) remove Postulates 4&5, replacing them with the following Postulate 4'.

Postulate 4'. The final move of a game is the one that produces a draw under any of the FIDE rules stated in Article 9, rules 9.2-9.3-9.6.

Let us define

$$Tot\ Draw := final\ score\ White\ (game\ ended) + \text{final}\ score\ Black\ (game\ ended), \quad (4)$$

asking which is the maximum value of $Tot\ Draw$, assuming Postulates 1 to 3 and Postulate 4'.

Lemma 4. The solution of Problem 5 can be easily proved to be 182 points (see [2], post #1), considering the ceiling given by (3) for both players minus the minimum amount of pieces needed to let any of the pawns pass and promote to a Queen, multiplying each of those pieces for its value (as stated in Postulate 1).
Thus, an upper bound of (4) is given by

$$Max\ Tot\ Draw\ = 2 \cdot 103 - 2 \cdot (2 \cdot N) - 2 \cdot (2 \cdot B) = 182. \quad (5)$$

Since 182 has been proved in [2] to be a score achievable playing only legit moves, we aim to solve the related stalemate problem, under the standard postulates 1 to 5 (as stated in Section 1), and including Constraint 1 too.

Problem 6. Which is the maximum value of (4) to let a standard chess game end in a stalemate?

Lemma 5. If we force the match to end with a stalemate, the highest possible value of (4) belongs to [168, 180].

Proof of Lemma 5.
$Max\ Tot\ Draw \leq 180$ naturally follows from Lemma 4, since the position shown in [2] can be easily claimed as a draw by rules 9.2&9.3, and it is trivial to point out how it is impossible to turn it into a stalemate (no stalemate with a queen for both players without a knight, a bishop or a pawn). So, we have: $Max\ Tot\ Draw\ \leq 182 - (R - B) = 182 - (5 - 3)$.
On the other hand, we can achieve a legit $Tot\ Draw$ of 168 points and a stalemate, as explained in [6], and subsequently confirmed by the standard match below.

PGN (1/2 - 1/2)
1. a4 b5 2. axb5 c5 3. d4 c4 4. Bd2 c3 5. d5 Nc6 6. dxc6 Ba6 7. b6 d5 8. b7 f5 9. Bf4 h5 10. Bd6 exd6 11. Nd2 g5 12. b8=Q Rh7 13. f4 cxd2+ 14. Kf2 Rd7 15. c7 d4 16. e4 d3 17. Qe2 d5 18. c8=Q d4 19. b4 Kf7 20. b5 Kg6 21. b6 Nf6 22. b7 Nd5 23. Qbc7 Nb4 24. b8=Q d1=Q 25. e5 d2 26. e6 d3 27. e7 Qb1 28. e8=Q+ Kf6 29. c4 d1=Q 30. c5 d2 31. c6 Qdb3 32. h3 Bb7 33. Qa5 Rd5 34. c7 Qe7 35. Qcd8 Re5 36. c8=Q Bd5 37. Qdb6+ Re6 38. Q2b5 d1=Q 39. Qb8b7 Nc6 40. fxg5+ Ke5 41. g6 f4 42. g4 Qd6 43. g5 Qba3 44. Be2 Qdb3 45. Bg4 hxg4 46. g7 f3 47. g8=Q Q1a2+ 48. Kg3 f2+ 49. Kh2 Qdb4 50. Ne2 f1=Q 51. Rc1 Nd4 52. h4 Rb8 53. Rc7 Qfa1 54. h5 g3+ 55. Kh3 g2+ 56. Kh4 g1=Q 57. h6 Qgb1 58. h7 Qb3b2 59. g6 Ke4 60. h8=Q Nb3 61. Qbd7 Bh6 62. g7 Bc1

63. Qgf8 Nd2 64. g8=Q Q4b3 65. Qac5 Kd3 66. Ng3 Re4+ 67. Qdg4 Rc4 68. Qed8 Rc2 69. Qde8 Bc4 70. Qhh7+ Kc3 71. Qbe6 a5 72. Qba6 a4 73. Qaa5+ Rb4 74. Qe1 □

The final position is shown in figure 5
(FEN: 2Q1QQQ1/2R4Q/8/Q1Q5/prb3QK/qqk3N1/qqrn4/qqb1Q2R b - 3 74).

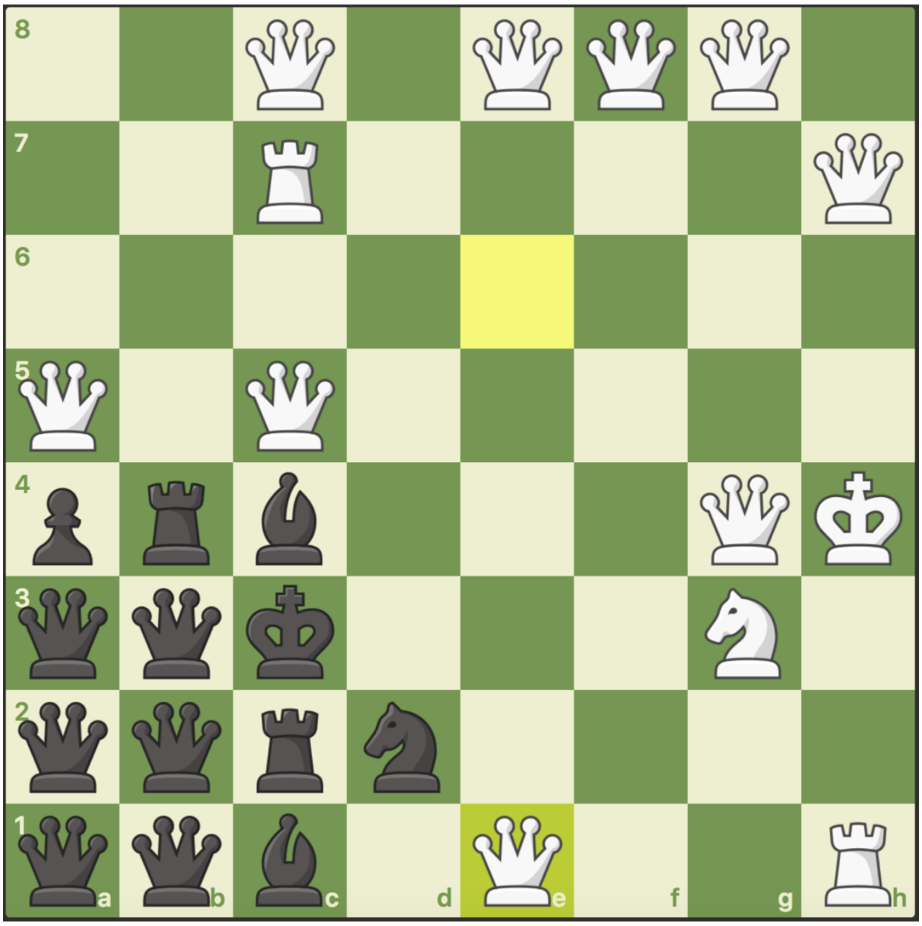

Figure 5. Final position of a stalemate for $Max\ Tot\ Draw = 168$.

4 Conclusion and open problems

Problems 1&2 have been definitively solved at the beginning of Section 2, while the last questions (Problems 3-4-6) still need to receive a final answer. The nontrivial bounds we have proved do not represent a solid celing for these hard open problems that we are going to further investigate in the future, in spite of we conjecture that the maximum value of (4) for a regular stalemate is 168.

5 Acknowledgements

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